A NOTE ON QUANTUM TELEPORTATION

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Since there was some confusion about how the quantum teleportation protocol (appeared originally in [1]) works during the lecture on April 1., the present note attempts to offer a more detailed perspective than the one found in the introductory chapter of *Categories for Quantum Theory*. The interested reader is also encouraged to consult standard texts on the subject, in particular [2].

1. The Bell basis

The quantum teleportation protocol is a way of transfering the state of a quantum bit by (a) consuming one half of an entangled pair (this is done by measuring it, and must be considered as consuming that resource since measurement breaks entanglement), and (b) using classical means to transfer two bits of classical information. At the core, the protocol exploits the fact that the four *Bell states* serve as a basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$. The Bell states are

$$\begin{aligned}
|\Phi^+\rangle &= \tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
|\Phi^-\rangle &= \tfrac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \\
|\Psi^+\rangle &= \tfrac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \\
|\Psi^-\rangle &= \tfrac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)
\end{aligned}$$

To see that these form a basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$, it suffices to show that we can derive the *computational basis* (i.e., $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$) as a linear combination of these. For example, we have that

$$\begin{aligned}
\tfrac{1}{\sqrt{2}}\left(|\Phi^+\rangle + |\Phi^-\rangle\right) &= \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) + \tfrac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)\right) \\
&= \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle + \tfrac{1}{\sqrt{2}}|00\rangle - \tfrac{1}{\sqrt{2}}|11\rangle\right) \\
&= \tfrac{1}{\sqrt{2}}\left(\tfrac{2}{\sqrt{2}}|00\rangle\right) = \tfrac{2}{2}|00\rangle = |00\rangle \qquad (1)
\end{aligned}$$

Similarly, one can show that

$$\tfrac{1}{\sqrt{2}}\left(|\Phi^+\rangle - |\Phi^-\rangle\right) = |11\rangle \qquad (2)$$

$$\tfrac{1}{\sqrt{2}}\left(|\Psi^+\rangle + |\Psi^-\rangle\right) = |01\rangle \qquad (3)$$

$$\tfrac{1}{\sqrt{2}}\left(|\Psi^+\rangle - |\Psi^-\rangle\right) = |10\rangle \qquad (4)$$

That the Bell states form a basis is crucial for quantum teleportation, since it allows us to *measure* states in $\mathbb{C}^2 \otimes \mathbb{C}^2$ in this basis.

2. Quantum teleportation

Armed with the Bell basis and equations (1)–(4), we are ready to discuss the actual teleportation protocol. The setup is as follows: Alice has a qubit that she wants to transfer to Bob, as well as one half of a maximally entangled state. Bob has the other half of the maximally entangled state. Recall that the protocol then proceeds as follows:

(i) Alice measures here qubit and her half of the entangled state in the Bell basis, yielding either $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, or $|\Psi^-\rangle$.
(ii) Alice communicates her measurement result to Bob by classical means. There are four possible outcomes, so this only requires two bits of information.

(iii) Based on the measurement outcome communicated by Alice, Bob performs "error correction" on his qubit by applying one of four unitary maps to it.
(iv) Bob's qubit is now in the same state as Alice's qubit was initially.

3. Why it works

To see why this protocol works, and what these unitary error correction maps must be, let Alice's qubit be in the state $\alpha|0\rangle + \beta|1\rangle$ for some coefficients α and β. The state of the entire system can be written in the computational basis as

$$(\alpha|0\rangle + \beta|1\rangle) \otimes \left(\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right) = \tfrac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

Rewriting this to isolate the first two systems (i.e., Alice's qubit and her half of the entangled pair), using the property of the tensor product that $s(|\phi\rangle \otimes |\psi\rangle) = (s|\phi\rangle) \otimes |\psi\rangle = |\phi\rangle \otimes (s|\psi\rangle)$ for any scalar s,

$$\begin{aligned}
&\tfrac{1}{\sqrt{2}}(\alpha|000\rangle + \beta|100\rangle + \alpha|011\rangle + \beta|111\rangle) \\
&\quad = \tfrac{1}{\sqrt{2}}((|00\rangle \otimes (\alpha|0\rangle)) + (|10\rangle \otimes (\beta|0\rangle)) + (|01\rangle \otimes (\alpha|1\rangle)) + (|11\rangle \otimes (\beta|1\rangle)))
\end{aligned}$$

Applying equations (1)–(4), isolating scalars, and using that tensors distribute over addition then gives us

$$\begin{aligned}
&\tfrac{1}{\sqrt{2}}((|00\rangle \otimes (\alpha|0\rangle)) + (|01\rangle \otimes (\alpha|1\rangle)) + (|10\rangle \otimes (\beta|0\rangle)) + (|11\rangle \otimes (\beta|1\rangle))) \\
&= \tfrac{1}{\sqrt{2}}(((\tfrac{1}{\sqrt{2}}\left(|\Phi^+\rangle + |\Phi^-\rangle\right)) \otimes (\alpha|0\rangle)) \ + ((\tfrac{1}{\sqrt{2}}\left(|\Psi^+\rangle + |\Psi^-\rangle\right)) \otimes (\alpha|1\rangle)) + \\
&\qquad ((\tfrac{1}{\sqrt{2}}\left(|\Psi^+\rangle - |\Psi^-\rangle\right)) \otimes (\beta|0\rangle)) \ + ((\tfrac{1}{\sqrt{2}}\left(|\Phi^+\rangle - |\Phi^-\rangle\right)) \otimes (\beta|1\rangle))) \\
&= \tfrac{1}{2}((|\Phi^+\rangle + |\Phi^-\rangle) \otimes (\alpha|0\rangle) + (|\Psi^+\rangle + |\Psi^-\rangle) \otimes (\alpha|1\rangle) + \\
&\qquad (|\Psi^+\rangle - |\Psi^-\rangle) \otimes (\beta|0\rangle) + (|\Phi^+\rangle - |\Phi^-\rangle) \otimes (\beta|1\rangle)) \\
&= \tfrac{1}{2}((|\Phi^+\rangle \otimes (\alpha|0\rangle)) + (|\Phi^-\rangle \otimes (\alpha|0\rangle)) + (|\Psi^+\rangle \otimes (\alpha|1\rangle)) + (|\Psi^-\rangle \otimes (\alpha|1\rangle)) + \\
&\qquad (|\Psi^+\rangle \otimes (\beta|0\rangle)) - (|\Psi^-\rangle \otimes (\beta|0\rangle)) + (|\Phi^+\rangle \otimes (\beta|1\rangle))) - (|\Phi^-\rangle \otimes (\beta|1\rangle)))) \\
&= \tfrac{1}{2}((|\Phi^+\rangle \otimes (\alpha|0\rangle)) + (|\Phi^-\rangle \otimes (\alpha|0\rangle)) + (|\Psi^+\rangle \otimes (\alpha|1\rangle)) + (|\Psi^-\rangle \otimes (\alpha|1\rangle)) + \\
&\qquad (|\Psi^+\rangle \otimes (\beta|0\rangle)) + (|\Psi^-\rangle \otimes (-\beta|0\rangle)) + (|\Phi^+\rangle \otimes (\beta|1\rangle))) + (|\Phi^-\rangle \otimes (-\beta|1\rangle)))) \qquad (5) \\
&= \tfrac{1}{2}((|\Phi^+\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)) + (|\Phi^-\rangle \otimes (\alpha|0\rangle - \beta|1\rangle)) + \\
&\qquad (|\Psi^+\rangle \otimes (\alpha|1\rangle + \beta|0\rangle)) + (|\Psi^-\rangle \otimes (\alpha|1\rangle - \beta|0\rangle))) \qquad (6)
\end{aligned}$$

In (5), we use the scalar isolation property of tensor products to move the outer -1 into the right-hand part of tensor products.

Up until this point, we haven't actually performed any steps in the protocol yet – we have simply rewritten the initial state into one where Alice's qubit and her half of the maximally entangled state are expressed in the Bell basis. However, looking at the state in (6) is nevertheless instructive, because it tells us precisely what state Bob's qubit must be in after Alice measures her qubit and her half of the entangled pair in the Bell basis:

(i) If Alice's measurement results in $|\Phi^+\rangle$, Bob's qubit must be in state $(\alpha|0\rangle + \beta|1\rangle)$. Since this was precisely the state of Alice's qubit to begin with, no error correction is necessary, so the unitary that Bob must apply in this case is simply the identity.
(ii) If Alice's measurement results in $|\Phi^-\rangle$, Bob's qubit must be in state $(\alpha|0\rangle - \beta|1\rangle)$. But in this case, Bob can recover the state of Alice's qubit by applying the unitary $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to his qubit, as $(\alpha|0\rangle + \beta|1\rangle) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, so

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\alpha|0\rangle - \beta|1\rangle) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|0\rangle + \beta|1\rangle$$

(iii) If Alice's measurement results in $|\Psi^+\rangle$, Bob's qubit must be in the state $\alpha|1\rangle + \beta|0\rangle$. Bob can recover the state of Alice's qubit by applying the NOT gate (or *x-rotation*) to his qubit, since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\alpha|1\rangle + \beta|0\rangle) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|0\rangle + \beta|1\rangle$$

(iv) Finally, if Alice's measurement results in $|\Psi^-\rangle$, Bob's qubit is in state $\alpha|1\rangle - \beta|0\rangle$. As such, applying both the unitary from (ii) and the one from (iii) lets Bob recover Alice's state, since

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\alpha|1\rangle - \beta|0\rangle) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|0\rangle + \beta|1\rangle. \end{aligned}$$

This shows the correctness of the protocol: No matter which state Alice observes after measuring her qubit and her half of the entangled pair in the Bell basis, she can classically communicate to Bob which one of four *predetermined* unitaries he must apply to his qubit in order to recover the state of Alice's qubit.

This is associated with a real resource cost, however, as Alice's measurement of her half of the entangled state breaks the entanglement. Even though Alice can presumably reuse the classical channel used to communicate the error correction term, every qubit "teleported" this way requires the expenditure of one maximally entangled pair.

References

[1] Bennett, C. H., Brassard, G., Crépeau, C., Jozsa, R., Peres, A. Wootters, W. K. *Teleporting an Unknown Quantum State via Dual Classical and Einstein–Podolsky–Rosen Channels.* Physical Review Letters **70** (13): 1895–1899, 1993.

[2] Nielsen, M. A. and Chuang, I. L., *Quantum Computation and Quantum Information* (10th Anniversary Edition), Cambridge University Press, 2010.